universität innsbruck

Hidden Markov Models

703075. Machine Learning

https://iis.uibk.ac.at/courses/2020s/703075

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Hidden Markov models

Introduction

Discrete Markov process

Observable Markov model

Hidden Markov model (HMM)

Solving HMMs

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Introduction

- Modeling dependencies in input; no longer iid
- Sequences:
  - Temporal: In speech; phonemes in a word (dictionary), words in a sentence (syntax, semantics of the language).
    - In handwriting, pen movements
  - Spatial: In a DNA sequence; base pairs

Introduction

- Modeling dependencies in input; no longer iid
- Sequences:
  - A sequence can be characterized as being generated by a *parametric random process*

Hidden Markov models

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Discrete Markov Process

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- Initial probabilities

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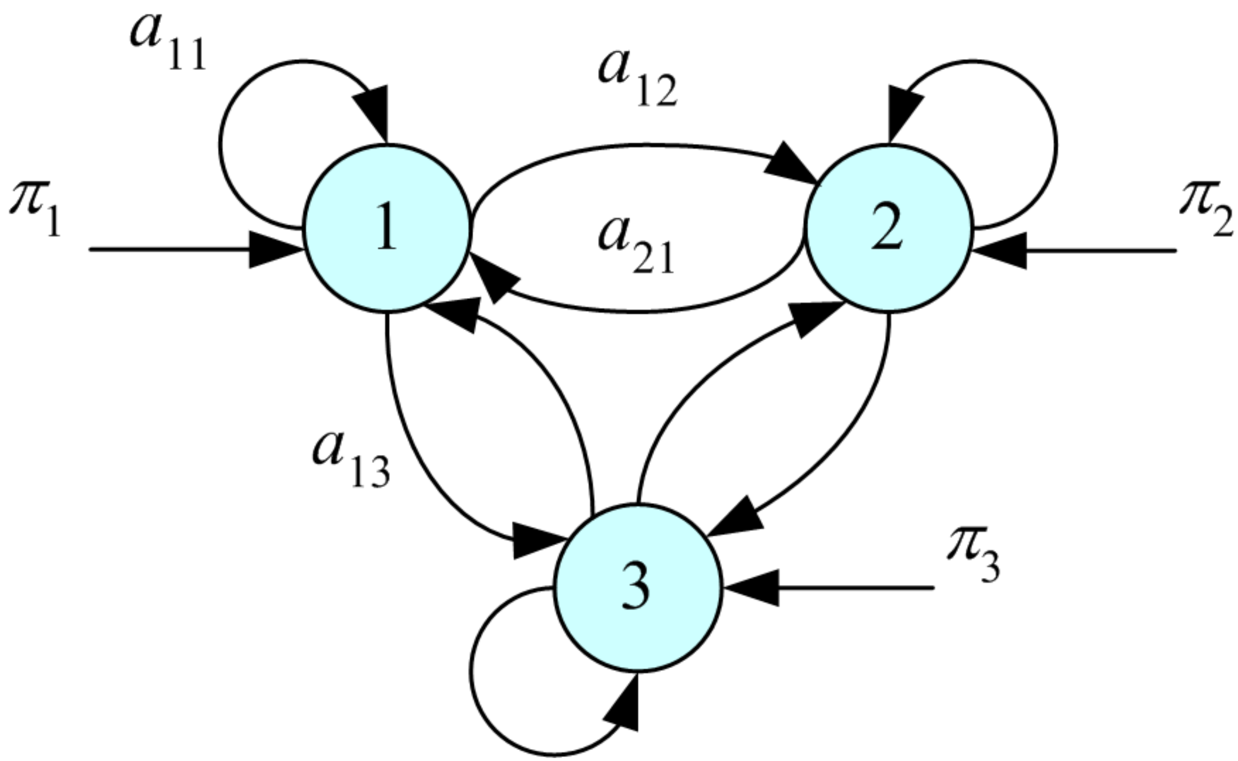
Stochastic Automation



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Observable Markov model

- The states are observable
  - At any time t we know q_t
  - Having an observation sequence

$$O = Q = \{q_1 q_2 \,\ldots.\, q_T\}$$

$$P(O = Q|\mathbf{A}, \mathbf{\Pi}) = P(q_1) \prod_{t=2}^{T} P(q_t|q_{t-1}) = \pi_{q_1} a_{q_1 q_2} \cdots a_{q_{T-1} q_T}$$

Example: Balls and Urns

- Rabiner and Juang (1986)
- Three urns each full of balls of one color

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$$P(O \mid \mathbf{A}, \Pi)$$

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$$P(O \mid \mathbf{A}, \Pi) = P(S_1) \cdot P(S_1 \mid S_1) \cdot P(S_3 \mid S_1) \cdot P(S_3 \mid S_3)$$

$$= \pi_1 \cdot a_{11} \cdot a_{13} \cdot a_{33}$$

$$= 0.5 \cdot 0.4 \cdot 0.3 \cdot 0.8 = 0.048$$

Example: Balls and Urns

- Learning
  - Given K example sequences of length T

$$\hat{\pi}_i = \frac{\#\{\text{sequences starting with } S_i\}}{\#\{\text{sequences}\}} = \frac{\sum_k 1\left(q_1^k = S_i\right)}{K}$$

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$$\hat{a}_{ij} = \frac{\#\{\text{transitions from } S_i \text{ to } S_j\}}{\#\{\text{transitions from } S_i\}}$$

$$= \frac{\sum_k \sum_{t=1}^{T\text{-}1} 1\left(q_t^k = S_i \text{ and } q_{t+1}^k = S_j\right)}{\sum_k \sum_{t=1}^{T\text{-}1} 1\left(q_t^k = S_i\right)}$$

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  - *Observation* that we observe
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- The state sequence Q is not observed
  - but it should be inferred from the observation sequence O

Example: Balls and Urns

- In each urn, there are balls of different colors, but with different probabilities.
  - For each observation sequence, there are multiple state sequences
  - $b_j(m) \equiv P(O_t=v_m \mid q_t=S_j)$ denotes the probability of drawing a ball of color m from urn j

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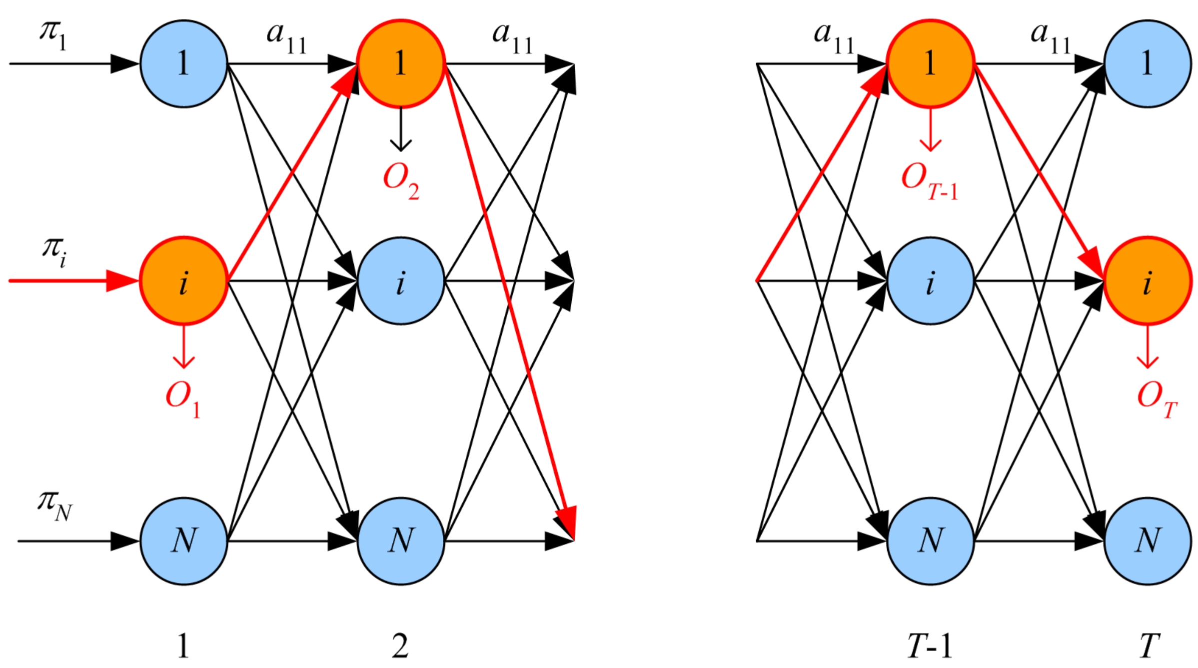
  - $b_j(m) \equiv P(O_t = v_m \mid q_t = S_j)$ denotes the probability of drawing a ball of color m from urn j
  - We again observe a sequence of ball colors but without knowing the sequence of urns from which the balls were drawn

Example: Balls and Urns

- In each urn, there are balls of different colors, but with different probabilities.
  - The observable model is a special case of the hidden model
  - where $M = N$
  - and $b_j(m)$ is 1 if $j = m$ and 0 otherwise

Example: Balls and Urns

- In each urn, there are balls of different colors, but with different probabilities.
  - For the same observation sequence O, there may be many possible state sequences Q that could have generated O



Elements of an HMM

- N: Number of states $S = \{S_1, S_2, \ldots, S_N\}$
- M: Number of observation symbols

$$V = \{v_1, v_2, \ldots, v_M\}$$

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- $\mathbf{A} = [a_{ij}]$: N by N state transition probability matrix

$$\mathbf{A} = [a_{ij}] \text{ where } a_{ij} \equiv P(q_{t+1} = S_j | q_t = S_i)$$

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$\lambda = (\mathbf{A}, \mathbf{B}, \mathbf{\Pi})$, parameter set of HMM

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     Given λ, and O, calculate $P(O \mid \lambda)$

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     Given λ, and O, find Q^* such that

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  3. Learning

     Given $\mathrm{X} = \{O^k\}_k$, find λ^* such that

     $$P(\mathrm{X} \mid \lambda^*) = \max_\lambda P(\mathrm{X} \mid \lambda)$$

1. Evaluation

- Given an observation sequence $O = \{O_1\, O_2 \cdots O_T\}$ and a state sequence $Q = \{q_1 q_2 \cdots q_T\}$, the probability of observing O given the state sequence Q is

$$P(O|Q,\lambda) = \prod_{t=1}^{T} P(O_t|q_t,\lambda) = b_{q_1}(O_1) \cdot b_{q_2}(O_2) \cdots b_{q_T}(O_T)$$

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- But, we can compute the probability of a state sequence $P(Q|\lambda) = P(q_1)\prod_{t=2}^{T} P(q_t|q_{t-1}) = \pi_{q_1} a_{q_1 q_2} \cdots a_{q_{T-1} q_T}$

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$$P(O|\lambda) = \sum_{\text{all possible } Q} P(O,Q|\lambda)$$

- There are N^T possible Q

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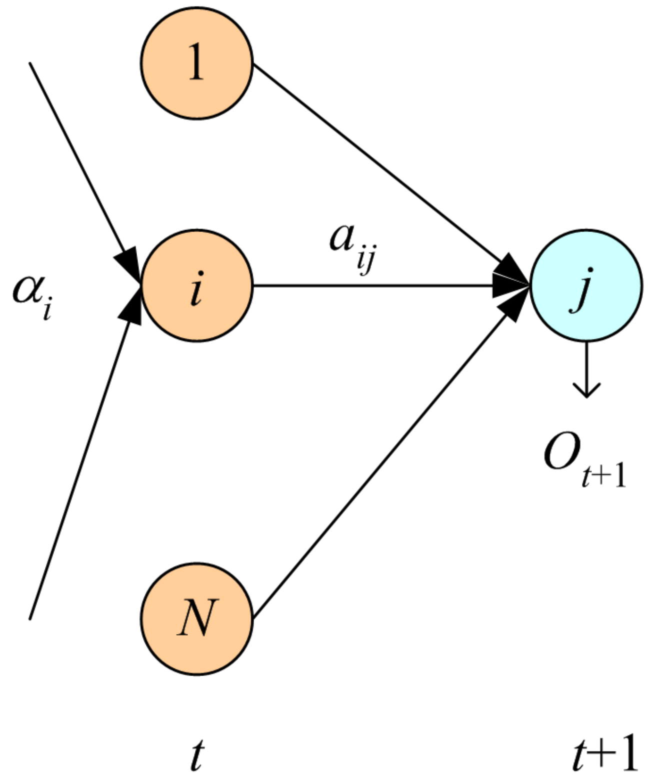
$$P(O|\lambda) = \sum_{i=1}^{N} P(O, q_T = S_i|\lambda)$$

- Forward-Backward procedure
  - Forward variable

$$\alpha_t(i) \equiv P(O_1 \cdots O_t, q_t = S_i \mid \lambda)$$



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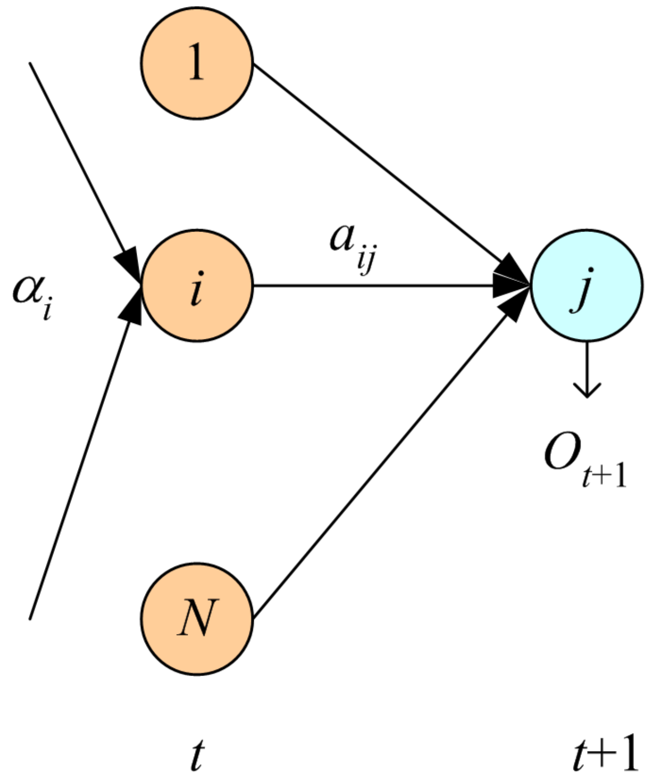

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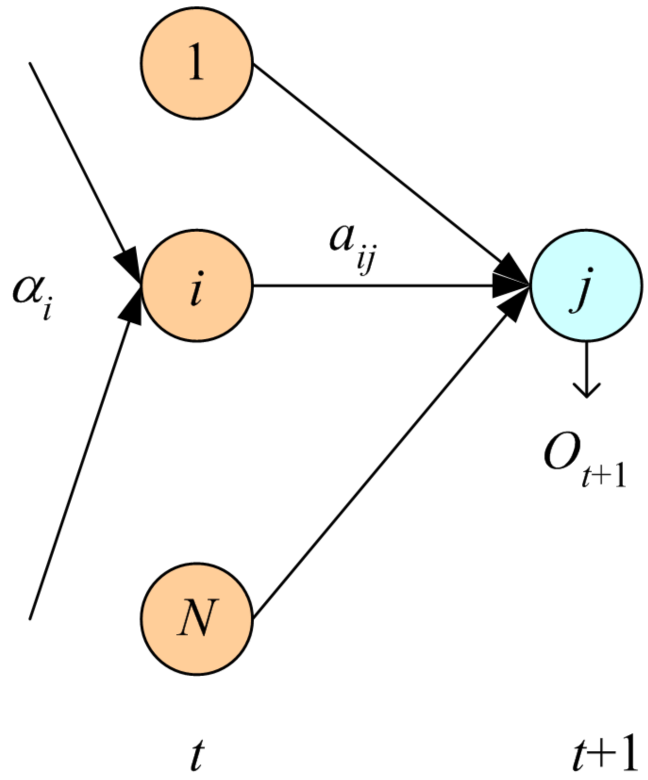

$$\begin{aligned}
\alpha_1(i) &\equiv P(O_1, q_1 = S_i|\lambda) \\
&= P(O_1|q_1 = S_i, \lambda)P(q_1 = S_i|\lambda) \\
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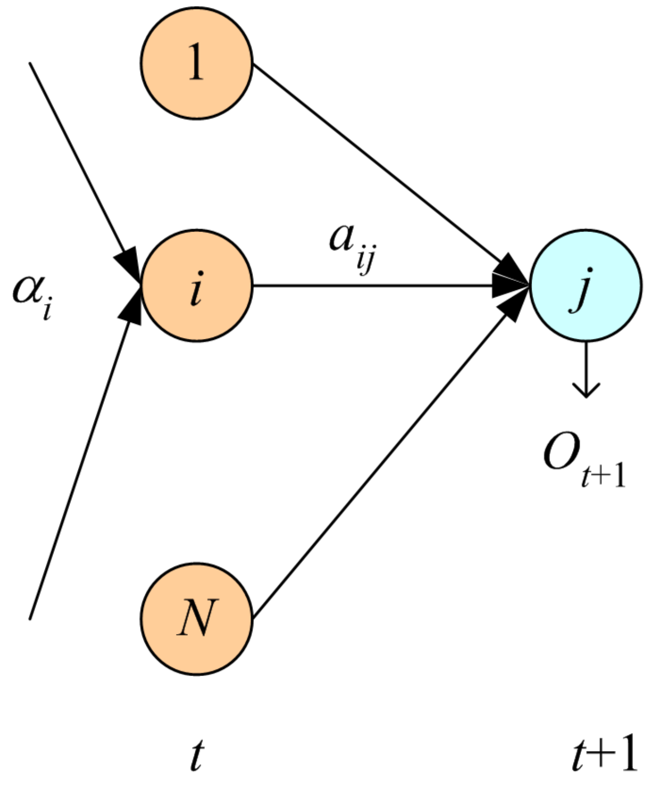
Recursion:

$$\alpha_{t+1}(j) = \left[\sum_{i=1}^{N} \alpha_t(i) a_{ij}\right] b_j(O_{t+1})$$



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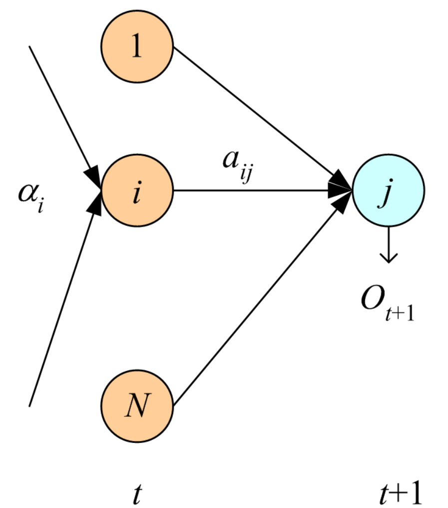

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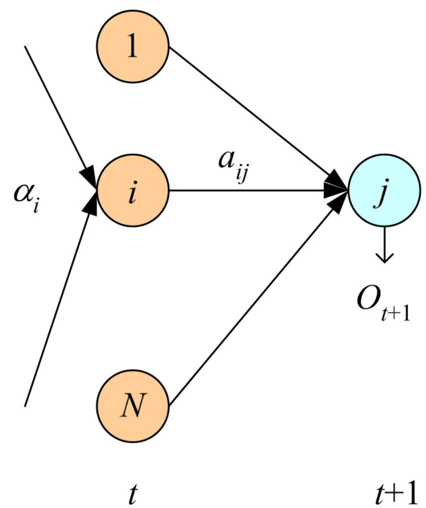

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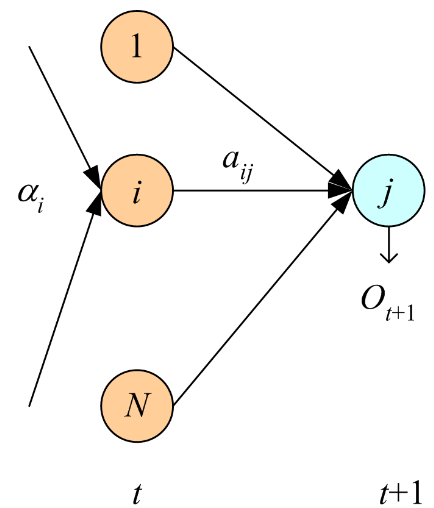

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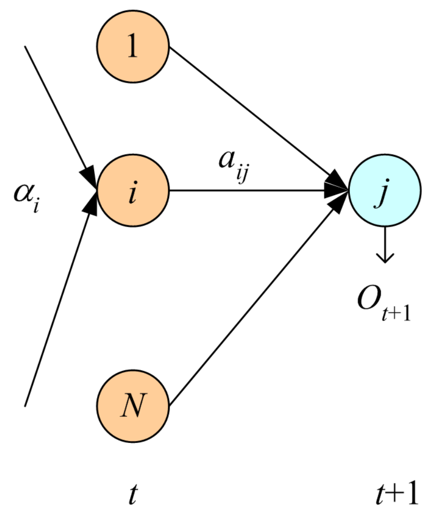

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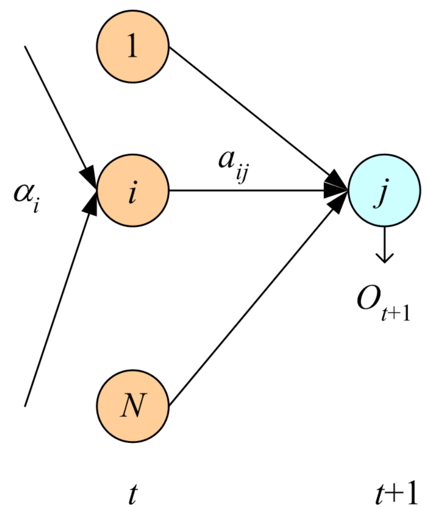

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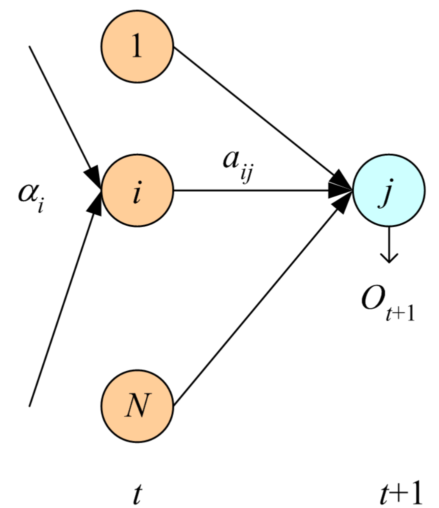

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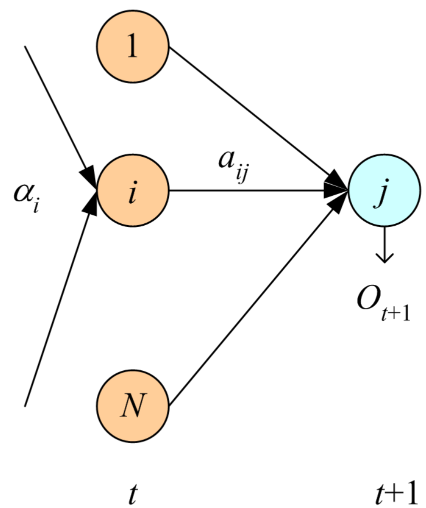
&= P(O_1 \cdots O_{t+1}|q_{t+1} = S_j, \lambda)P(q_{t+1} = S_j|\lambda) \\
&= P(O_1 \cdots O_t|q_{t+1} = S_j, \lambda)P(O_{t+1}|q_{t+1} = S_j, \lambda)P(q_{t+1} = S_j|\lambda) \\
&= P(O_1 \cdots O_t, q_{t+1} = S_j|\lambda)P(O_{t+1}|q_{t+1} = S_j, \lambda) \\
&= P(O_{t+1}|q_{t+1} = S_j, \lambda) \sum_i P(O_1 \cdots O_t, q_t = S_i, q_{t+1} = S_j|\lambda) \\
&= P(O_{t+1}|q_{t+1} = S_j, \lambda) \\
&\quad \sum_i P(O_1 \cdots O_t, q_{t+1} = S_j|q_t = S_i, \lambda)P(q_t = S_i|\lambda) \\
&= P(O_{t+1}|q_{t+1} = S_j, \lambda) \\
&\quad \sum_i P(O_1 \cdots O_t|q_t = S_i, \lambda)P(q_{t+1} = S_j|q_t = S_i, \lambda)P(q_t = S_i|\lambda)
\end{aligned}$$



1. Evaluation

$$P(O|\lambda) = \sum_{i=1}^{N} P(O, q_T = S_i|\lambda)$$

Recursion

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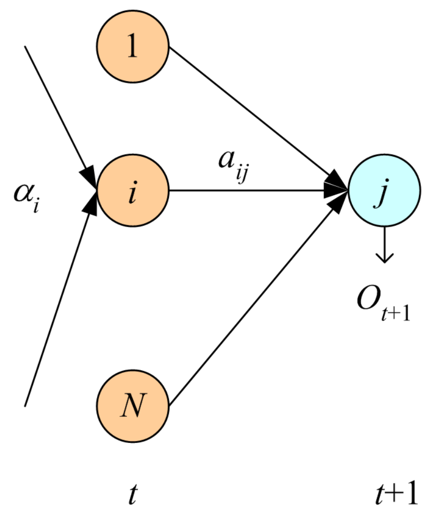

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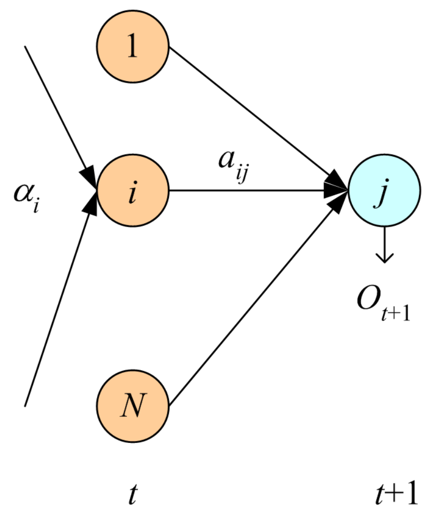

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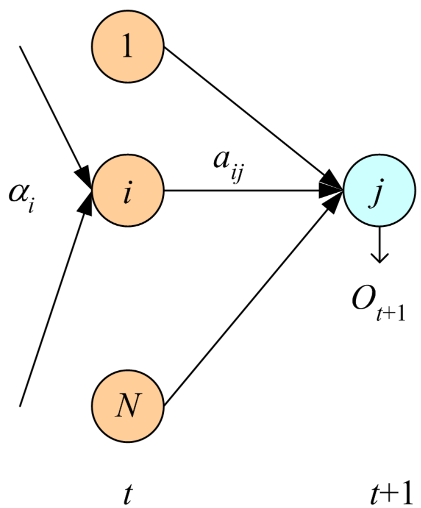
&\quad \sum_i P(O_1 \cdots O_t, q_t = S_i|\lambda) P(q_{t+1} = S_j|q_t = S_i, \lambda) \\
&= \left[\sum_{i=1}^{N} \alpha_t(i) a_{ij} \right] b_j(O_{t+1})
\end{aligned}$$



1. Evaluation

$$P(O|\lambda) = \sum_{i=1}^{N} P(O, q_T = S_i|\lambda)$$

- Forward-Backward procedure
  - Forward variable

$$\alpha_t(i) \equiv P(O_1 \cdots O_t, q_t = S_i \mid \lambda)$$

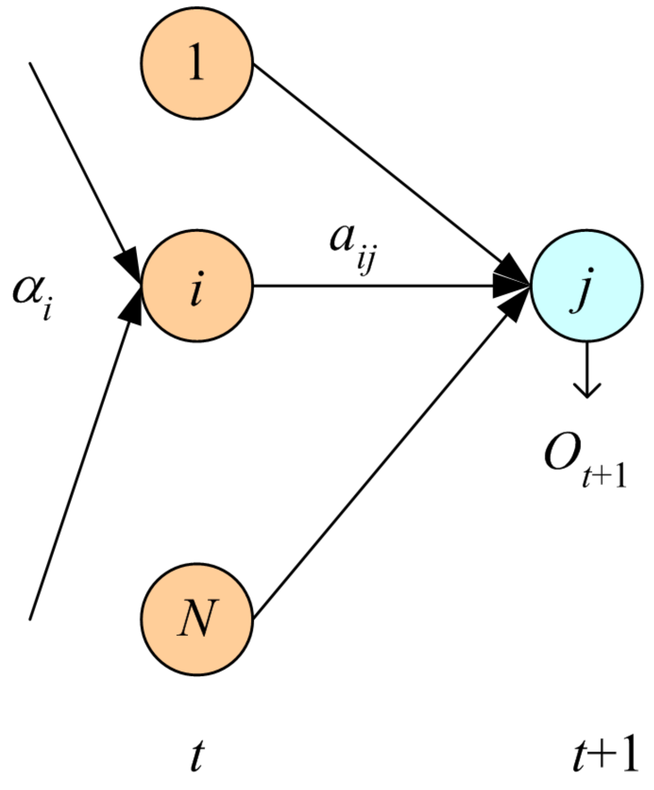
Initialization:

$$\alpha_1(i) = \pi_i b_i(O_1)$$

Recursion:

$$\alpha_{t+1}(j) = \left[\sum_{i=1}^{N} \alpha_t(i) a_{ij}\right] b_j(O_{t+1})$$

$$P(O \mid \lambda) = \sum_{i=1}^{N} \alpha_T(i)$$

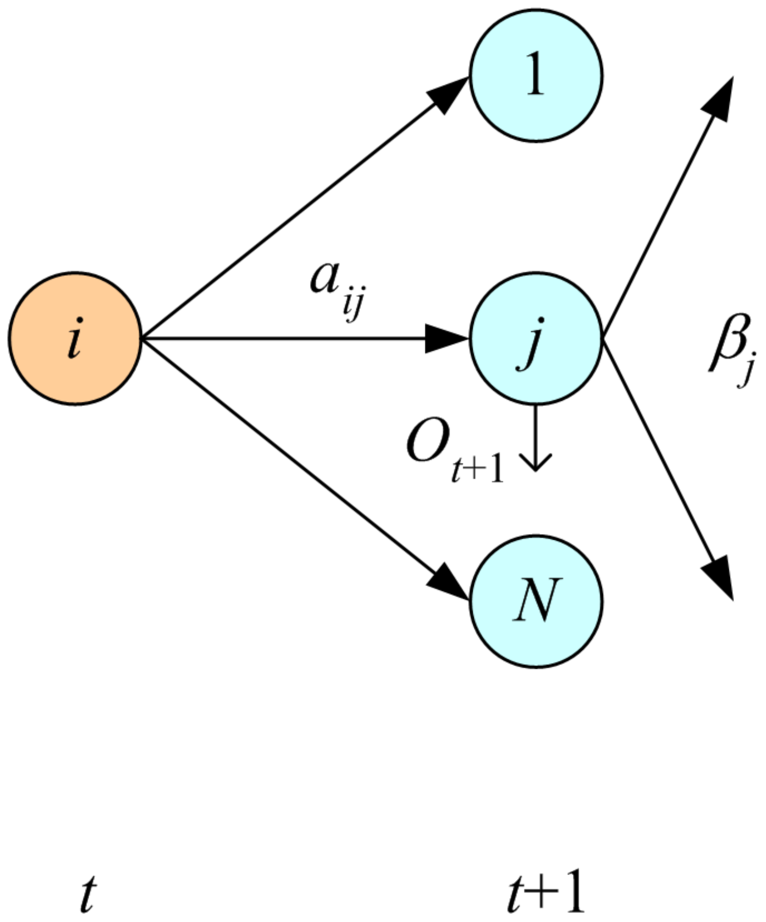


1. Evaluation

$$P(O|Q,\lambda) = \prod_{t=1}^{T} P(O_t|q_t,\lambda) = b_{q_1}(O_1)\cdot b_{q_2}(O_2)\cdots b_{q_T}(O_T)$$

- Forward-Backward procedure
  - Backward variable

$$\beta_t(i) \equiv P\left(O_{t+1}\cdots O_T \,|\, q_t = S_i, \lambda\right)$$



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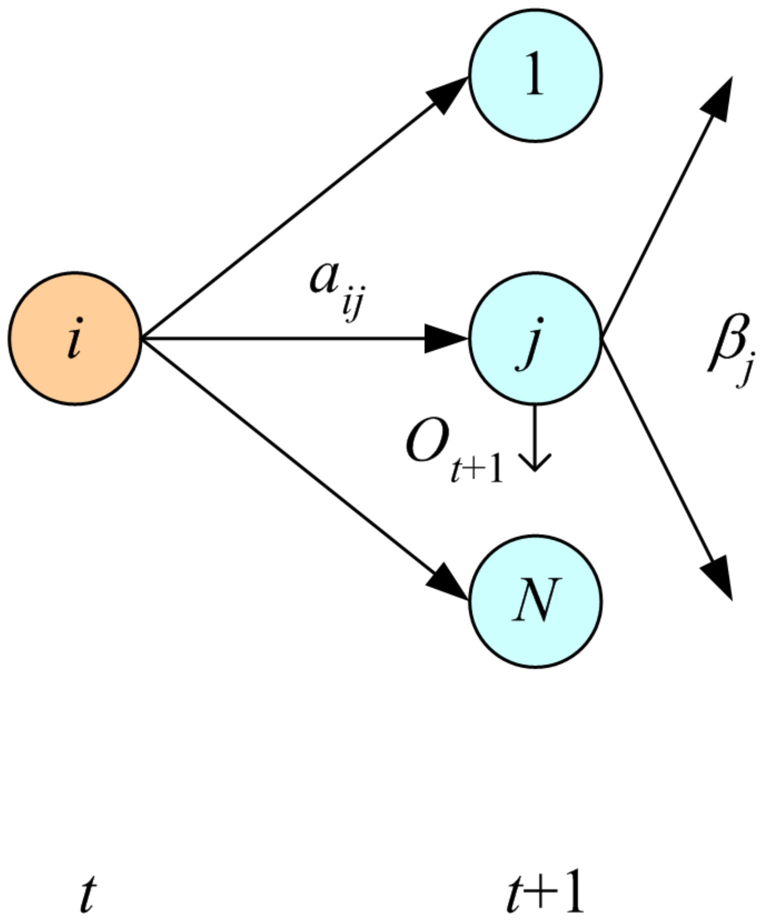
- Forward-Backward procedure
  - Backward variable

$$\beta_t(i) \equiv P(O_{t+1} \cdots O_T \mid q_t = S_i, \lambda)$$

Initialization:

$$\beta_T(i) = 1$$



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$$P(O|Q,\lambda) = \prod_{t=1}^{T} P(O_t|q_t,\lambda) = b_{q_1}(O_1) \cdot b_{q_2}(O_2) \cdots b_{q_T}(O_T)$$

- Forward-Backward procedure
  - Backward variable

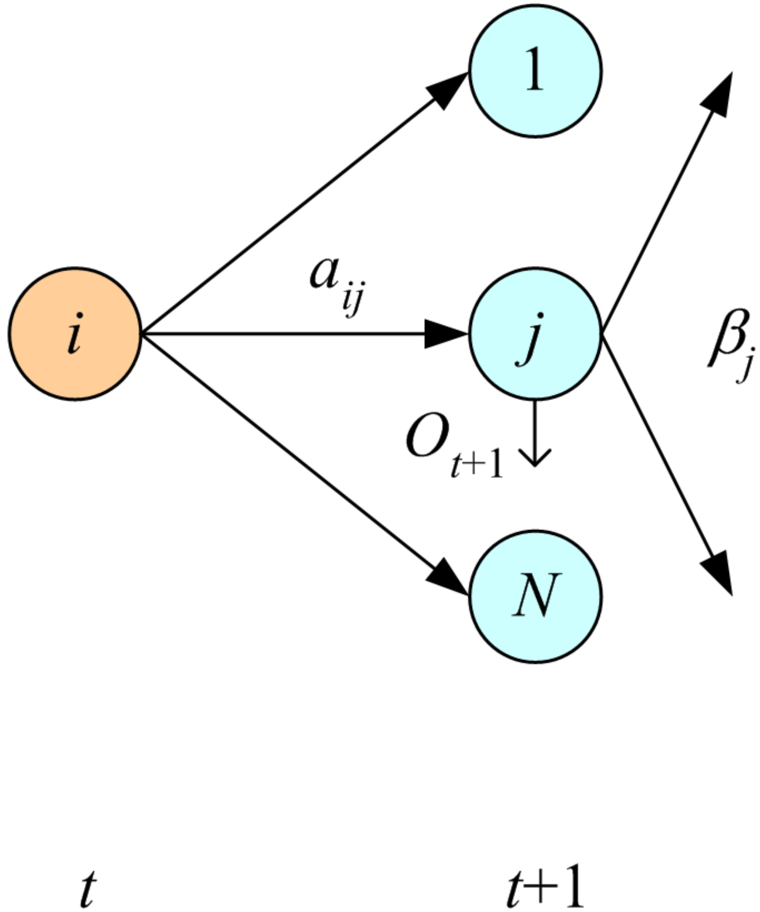
$$\beta_t(i) \equiv P(O_{t+1} \cdots O_T \mid q_t = S_i, \lambda)$$

Initialization:

$$\beta_T(i) = 1$$

Recursion:

$$\beta_t(i) = \sum_{j=1}^{N} a_{ij} b_j(O_{t+1}) \beta_{t+1}(j)$$



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$$\alpha_t(i) \equiv P(O_1 \cdots O_t, q_t = S_i \mid \lambda)$$

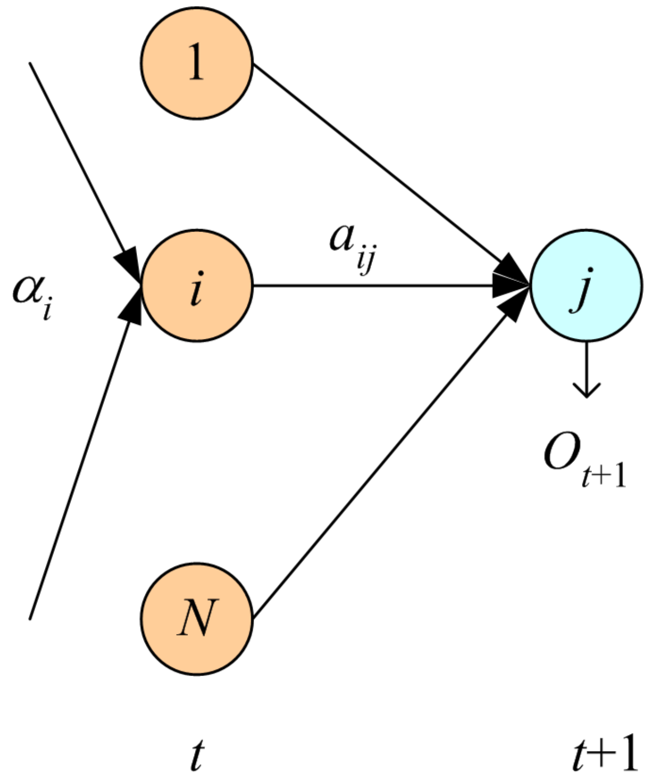
Initialization:

$$\alpha_1(i) = \pi_i b_i(O_1)$$

Recursion:

$$\alpha_{t+1}(j) = \left[\sum_{i=1}^{N} \alpha_t(i) a_{ij}\right] b_j(O_{t+1})$$

$$P(O \mid \lambda) = \sum_{i=1}^{N} \alpha_T(i)$$



2. State sequence

- Find the state sequence $Q = \{q_1 q_2 \cdots q_T\}$ having the highest probability of generating the observation sequence $O = \{O_1\ O_2 \cdots O_T\}$, given the model λ

$$P(Q^* \mid O, \lambda) = \max_Q P(Q \mid O, \lambda)$$

2. State sequence

$$P(Q^* \mid O, \lambda) = \max_Q P(Q \mid O, \lambda)$$

- Let us define $\gamma_t(i)$ as the probability of being in state S_i at time t, given O and λ

$$\begin{aligned} \gamma_t(i) &\equiv P(q_t = S_i | O, \lambda) \\ &= \frac{P(O | q_t = S_i, \lambda) P(q_t = S_i | \lambda)}{P(O | \lambda)} \end{aligned}$$

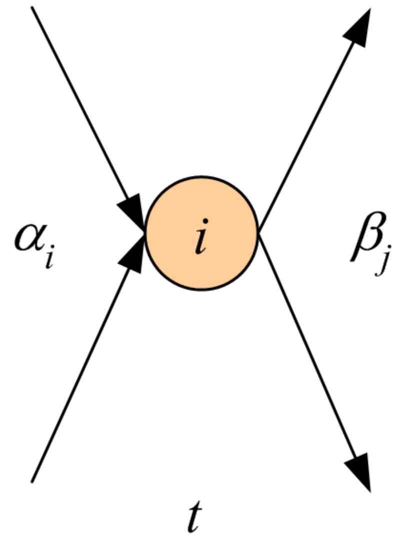
2. State sequence

$$P(Q^* \mid O, \lambda) = \max_Q P(Q \mid O, \lambda)$$

- Let us define $\gamma_t(i)$ as the probability of being in state S_i at time t, given O and λ

$$\gamma_t(i) \equiv P(q_t = S_i \mid O, \lambda)$$

$$= \frac{\alpha_t(i)\beta_t(i)}{\sum_{j=1}^{N} \alpha_t(j)\beta_t(j)}$$



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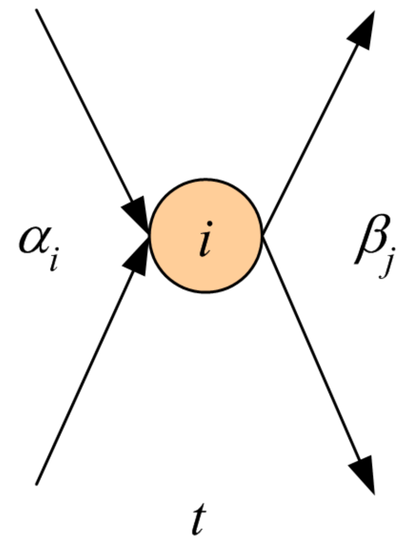

- Choose the state that has the highest probability
- for each time step: $q_t^* = \arg\max_i \gamma_t(i)$

2. State sequence

$$\gamma_t(i) \equiv P(q_t = S_i | O, \lambda)$$
$$= \frac{\alpha_t(i)\beta_t(i)}{\sum_{j=1}^{N} \alpha_t(j)\beta_t(j)}$$

- Choose the state that has the highest probability
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- **Viterbi Algorithm**

2. State sequence

$$\gamma_t(i) \equiv P(q_t = S_i | O, \lambda)$$
$$= \frac{\alpha_t(i)\beta_t(i)}{\sum_{j=1}^{N} \alpha_t(j)\beta_t(j)}$$

- Viterbi Algorithm
  - Given state sequence $Q = q_1 q_2 \cdots q_T$ and observation sequence $O = O_1 \cdots O_T$, we define $\delta_t(i)$ as the probability of the highest probability path at time t that accounts for the first t observations and ends in S_i

$$\delta_t(i) \equiv \max_{q_1 q_2 \cdots q_{t-1}} p(q_1 q_2 \cdots q_{t-1}, q_t = S_i, O_1 \cdots O_t | \lambda)$$

2. State sequence

$$\delta_t(i) \equiv \max_{q_1 q_2 \cdots q_{t-1}} p(q_1 q_2 \cdots q_{t-1}, q_t = S_i, O_1 \cdots O_t | \lambda)$$

- Viterbi Algorithm
  - Initialization:

$$\delta_1(i) = \pi_i b_i(O_1)$$

$$\psi_1(i) = 0$$

2. State sequence

$$\delta_t(i) \equiv \max_{q_1 q_2 \cdots q_{t-1}} p(q_1 q_2 \cdots q_{t-1}, q_t = S_i, O_1 \cdots O_t | \lambda)$$

- Viterbi Algorithm
  - Initialization:

$$\delta_1(i) = \pi_i b_i(O_1)$$

$$\psi_1(i) = 0$$

  - Recursion

$$\delta_t(j) = \max_i \delta_{t\text{-}1}(i) a_{ij} b_j(O_t)$$

$$\psi_t(j) = \mathrm{argmax}_i\, \delta_{t\text{-}1}(i) a_{ij}$$

2. State sequence

$$\delta_t(i) \equiv \max_{q_1 q_2 \cdots q_{t-1}} p(q_1 q_2 \cdots q_{t-1}, q_t = S_i, O_1 \cdots O_t | \lambda)$$

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  - Termination:

$$p^* = \max_i \delta_T(i)$$

$$q_T^* = \operatorname{argmax}_i \delta_T(i)$$

2. State sequence

$$\delta_t(i) \equiv \max_{q_1 q_2 \cdots q_{t-1}} p(q_1 q_2 \cdots q_{t-1}, q_t = S_i, O_1 \cdots O_t | \lambda)$$

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$$\psi_t(j) = \text{argmax}_i \delta_{t\text{-}1}(i) a_{ij}$$

  - Path bactracking

$$q_t^* = \psi_{t+1}(q_{t+1}^*), t=T\text{-}1, T\text{-}2, \ldots, 1$$

  - Termination:

$$p^* = \max_i \delta_T(i)$$

$$q_T^* = \text{argmax}_i \delta_T(i)$$

2. State sequence

$$\delta_t(i) \equiv \max_{q_1 q_2 \cdots q_{t-1}} p(q_1 q_2 \cdots q_{t-1}, q_t = S_i, O_1 \cdots O_t | \lambda)$$

- Viterbi Algorithm
  - Recursion

    $$\delta_t(j) = \max_i \delta_{t-1}(i) a_{ij} b_j(O_t)$$

    $$\psi_t(j) = \text{argmax}_i \delta_{t-1}(i) a_{ij}$$

  - Termination:

    $$p^* = \max_i \delta_T(i)$$

    $$q_T^* = \text{argmax}_i \delta_T(i)$$

  - Path bactracking

    $$q_t^* = \psi_{t+1}(q_{t+1}^*), t = T\text{-}1, T\text{-}2, \ldots, 1$$

  - $\psi_t(j)$ keeps track of the state that maximizes $\delta_t(j)$ at time $t-1$, that is, the best previous state

3. Learning

- Maximum Likelihood
  - Calculate $\lambda*$ that maximizes the likelihood of the sample of training sequences, $\mathrm{X} = \{O^k\}^K_{k=1}$, namely, $P(\mathrm{X}|\lambda)$

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3. Learning

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$$\begin{aligned}\xi_t(i,j) &\equiv P(q_t = S_i, q_{t+1} = S_j | O, \lambda) \\ &= \frac{P(O|q_t = S_i, q_{t+1} = S_j, \lambda) P(q_t = S_i, q_{t+1} = S_j | \lambda)}{P(O|\lambda)}\end{aligned}$$

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&= \frac{P(O|q_t = S_i, q_{t+1} = S_j, \lambda) P(q_t = S_i, q_{t+1} = S_j | \lambda)}{P(O|\lambda)} \\
&= \frac{P(O|q_t = S_i, q_{t+1} = S_j, \lambda) P(q_{t+1} = S_j | q_t = S_i, \lambda) P(q_t = S_i | \lambda)}{P(O|\lambda)} \\
&= \left(\frac{1}{P(O|\lambda)}\right) P(O_1 \cdots O_t | q_t = S_i, \lambda) P(O_{t+1} | q_{t+1} = S_j, \lambda) \\
&\quad P(O_{t+2} \cdots O_T | q_{t+1} = S_j, \lambda) a_{ij} P(q_t = S_i | \lambda)
\end{aligned}$$

3. Learning

- Maximum Likelihood

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&\quad P(O_{t+2} \cdots O_T|q_{t+1} = S_j, \lambda)a_{ij}P(q_t = S_i|\lambda) \\
&= \left(\frac{1}{P(O|\lambda)}\right) P(O_1 \cdots O_t, q_t = S_i|\lambda)P(O_{t+1}|q_{t+1} = S_j, \lambda) \\
&\quad P(O_{t+2} \cdots O_T|q_{t+1} = S_j, \lambda)a_{ij} \\
&= \frac{\alpha_t(i)b_j(O_{t+1})\beta_{t+1}(j)a_{ij}}{\sum_k \sum_l P(q_t = S_k, q_{t+1} = S_l, O|\lambda)}
\end{aligned}$$

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- Maximum Likelihood

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&= \left(\frac{1}{P(O|\lambda)}\right) P(O_1 \cdots O_t, q_t = S_i|\lambda) P(O_{t+1}|q_{t+1} = S_j, \lambda) \\
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&= \frac{\alpha_t(i) b_j(O_{t+1}) \beta_{t+1}(j) a_{ij}}{\sum_k \sum_l P(q_t = S_k, q_{t+1} = S_l, O|\lambda)} \\
&= \frac{\alpha_t(i) a_{ij} b_j(O_{t+1}) \beta_{t+1}(j)}{\sum_k \sum_l \alpha_t(k) a_{kl} b_l(O_{t+1}) \beta_{t+1}(l)}
\end{aligned}$$

3. Learning

- Maximum Likelihood

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- We can calculate the probability of being in state S_i at time t by marginalizing over the arc probabilities for all possible next states

$$\gamma_t(i) = \sum_{j=1}^{N} \xi_t(i,j)$$

3. Learning

- Maximum Likelihood
  - We can calculate the probability of being in state S_i at time t by marginalizing over the arc probabilities for all possible next states

$$\gamma_t(i) = \sum_{j=1}^{N} \xi_t(i,j)$$

  - If the Markov model were not hidden but observable, both $\gamma_t(i)$ and $\xi_t(i,j)$ would be 0/1.
  - In this case when they are not, we estimate them with posterior probabilities that give us *soft counts*

3. Learning

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  - We can calculate the probability of being in state S_i at time t by marginalizing over the arc probabilities for all possible next states

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  - In this case when they are not, we estimate them with posterior probabilities that give us *soft counts*
  - We estimate posterior probabilities first (in the E-step) and calculate the parameters with these estimates (in the M-step).

3. Learning

- Baum-Welch algorithm
  - E-step computes $\xi_t(i,j)$ and $\gamma_t(i)$
  - M-step re-calculate λ given $\xi_t(i,j)$ and $\gamma_t(i)$
  - Until convergence
    - $P(O|\lambda)$ never decreases

3. Learning

- Baum-Welch algorithm
  - E-step computes $\xi_t(i,j)$ and $\gamma_t(i)$

$$E[z_i^t] = \gamma_t(i)$$

$$E[z_{ij}^t] = \xi_t(i,j)$$

$$z_i^t = \begin{cases} 1 & \text{if } q_t = S_i \\ 0 & \text{otherwise} \end{cases} \qquad z_{ij}^t = \begin{cases} 1 & \text{if } q_t = S_i \text{ and } q_{t+1} = S_j \\ 0 & \text{otherwise} \end{cases}$$

$$\xi_t(i,j) = \frac{\alpha_t(i) a_{ij} b_j(O_{t+1}) \beta_{t+1}(j)}{\sum_k \sum_l \alpha_t(k) a_{kl} b_l(O_{t+1}) \beta_{t+1}(l)}$$

$$\gamma_t(i) = \sum_{j=1}^{N} \xi_t(i,j)$$

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- Baum-Welch algorithm
  - E-step computes $\xi_t(i,j)$ and $\gamma_t(i)$

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$$\xi_t(i,j) = \frac{\alpha_t(i) a_{ij} b_j(O_{t+1}) \beta_{t+1}(j)}{\sum_k \sum_l \alpha_t(k) a_{kl} b_l(O_{t+1}) \beta_{t+1}(l)} \qquad \gamma_t(i) = \sum_{j=1}^{N} \xi_t(i,j)$$

  - M-step re-calculate λ given $\xi_t(i,j)$ and $\gamma_t(i)$
    - Ratio of number of transitions from S_i to S_j

$$\hat{a}_{ij} = \frac{\sum_{t=1}^{T-1} \xi_t(i,j)}{\sum_{t=1}^{T-1} \gamma_t(i)}$$

    - Probability of observing v_m in S_j $\quad \hat{b}_j(m) = \dfrac{\sum_{t=1}^{T} \gamma_t(j) 1(O_t = v_m)}{\sum_{t=1}^{T} \gamma_t(j)}$

3. Learning

- Baum-Welch algorithm
  - E-step computes $\xi_t(i,j)$ and $\gamma_t(i)$

$$E[z_i^t] = \gamma_t(i)$$
$$E[z_{ij}^t] = \xi_t(i,j)$$

$$z_i^t = \begin{cases} 1 & \text{if } q_t = S_i \\ 0 & \text{otherwise} \end{cases} \qquad z_{ij}^t = \begin{cases} 1 & \text{if } q_t = S_i \text{ and } q_{t+1} = S_j \\ 0 & \text{otherwise} \end{cases}$$

$$\xi_t(i,j) = \frac{\alpha_t(i) a_{ij} b_j(O_{t+1}) \beta_{t+1}(j)}{\sum_k \sum_l \alpha_t(k) a_{kl} b_l(O_{t+1}) \beta_{t+1}(l)} \qquad \gamma_t(i) = \sum_{j=1}^{N} \xi_t(i,j)$$

  - M-step re-calculate λ given $\xi_t(i,j)$ and $\gamma_t(i)$
    - For multiple observation sequences

$$\hat{a}_{ij} = \frac{\sum_{k=1}^{K} \sum_{t=1}^{T_k-1} \xi_t^k(i,j)}{\sum_{k=1}^{K} \sum_{t=1}^{T_k-1} \gamma_t^k(i)} \qquad \hat{\pi}_i = \frac{\sum_{k=1}^{K} \gamma_1^k(i)}{K}$$

$$\hat{b}_j(m) = \frac{\sum_{k=1}^{K} \sum_{t=1}^{T_k} \gamma_t^k(j) 1(O_t^k = v_m)}{\sum_{k=1}^{K} \sum_{t=1}^{T_k} \gamma_t^k(j)}$$

Classification with HMMs

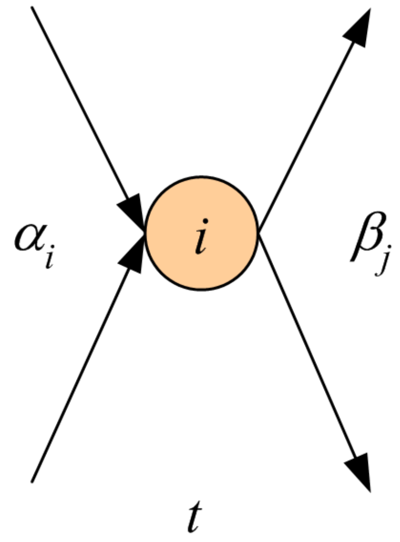
- We have a set of HMMs, each one modeling the sequences belonging to one class
  - For example, in spoken word recognition, examples of each word train a separate model, λ_i.
  - Given a new word utterance O to classify, all of the separate word models are evaluated to calculate $P(O|\lambda_i)$.
  - We then use Bayes' rule to get the posterior probabilities $$P(\lambda_i|O) = \frac{P(O|\lambda_i)P(\lambda_i)}{\sum_j P(O|\lambda_j)P(\lambda_j)}$$

Exercise 1

- Prove the recursion expression for the forward-backward algorithm:

Recursion:

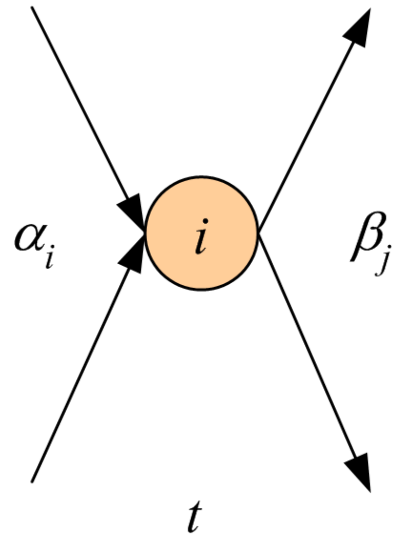
$$\beta_t(i) = \sum_{j=1}^{N} a_{ij} b_j(O_{t+1}) \beta_{t+1}(j)$$



Exercise 2

- Prove that for finding the state sequence

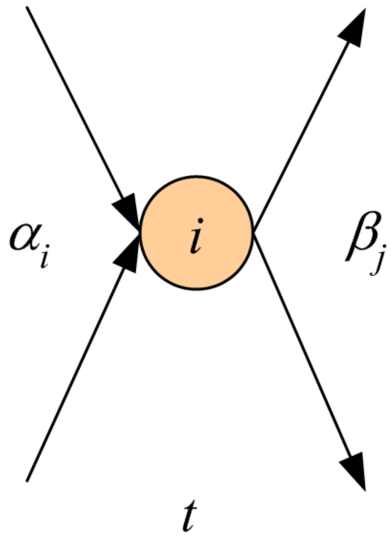
$$\gamma_t(i) \equiv P(q_t = S_i \mid O, \lambda) = \frac{\alpha_t(i)\beta_t(i)}{\sum_{j=1}^{N} \alpha_t(j)\beta_t(j)}$$



Exercise 2

- Prove that for finding the state sequence

$$\gamma_t(i) \equiv P(q_t = S_i | O, \lambda)$$
$$= \frac{\alpha_t(i)\beta_t(i)}{\sum_{j=1}^{N} \alpha_t(j)\beta_t(j)}$$



$$\gamma_t(i) \equiv P(q_t = S_i | O, \lambda)$$
$$= \frac{P(O | q_t = S_i, \lambda) P(q_t = S_i | \lambda)}{P(O | \lambda)}$$

Summary

Introduction

Discrete Markov process

Observable Markov model

Hidden Markov model (HMM)

Solving HMMs