Supplementary Material

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November 14, 2013

1 Computing Gradient

Most of computing in this section is based on the matrix calculus rules in (Facker, 2004).

$$\begin{aligned}\{\mathbf{R}^*, \mathbf{t}^*\} &= \arg\max_{\mathbf{R},\mathbf{t}} \mathbf{O} \\ &= \arg\max_{\mathbf{R},\mathbf{t}} K(\mathcal{M}_{\mathbf{R},\mathbf{t}}, \mathcal{S}) \\ &= \arg\max_{\mathbf{R},\mathbf{t}} \Big\{ \tfrac{1}{m_\mathcal{M}} \tfrac{1}{m_\mathcal{S}} \textstyle\sum_{i=1}^{m_\mathcal{M}} \sum_{j=1}^{m_\mathcal{S}} \tfrac{1}{(2\pi)^{3/2}} \underbrace{\frac{1}{|\mathbf{G}_{ij}|^{1/2}} \exp\left(-\frac{1}{2}{\Delta_{ij}}'{\mathbf{G}_{ij}}^{-1}\Delta_{ij}\right)}_{\mathcal{O}_{ij}} \Big\}\end{aligned}$$

where $\mathbf{G}_{ij} = \Sigma_\mathcal{S}^{(j)} + \mathbf{R}\Sigma_\mathcal{M}^{(i)}\mathbf{R}'$ and $\Delta_{ij} = \lambda_\mathcal{S}^{(j)} - \mathbf{R}\lambda_\mathcal{M}^{(i)} - \mathbf{t}$. By using the mapping:

$$\Omega : [\mathbf{w}, \mathbf{v}]' \to \{\exp(J(\mathbf{w})\mathbf{R}_0), \mathbf{t}_0 + \mathbf{v}\}, \quad \mathbf{w}, \mathbf{v}, \mathbf{t}_0 \in \mathbb{R}^3, \mathbf{R}_0 \in SO(3)$$

the objective function is dependent only on $\mathbf{w}$ and $\mathbf{v}$, and the full gradient can be computed as the sum of gradient on each pair of points. So for each pair point $\lambda_\mathcal{M}^{(i)}$ and $\lambda_\mathcal{S}^{(j)}$, the objective function is as follows:

$$\begin{aligned}\mathcal{O}_{ij} \quad &= \overbrace{|\Sigma_\mathcal{S}^{(j)} + \mathbf{R}\Sigma_\mathcal{M}^{(i)}\mathbf{R}'|^{-1/2}}^{|\mathbf{G}_{ij}|^{-1/2}} \cdot \\ &\quad \exp\Big\{ -\tfrac{1}{2} \underbrace{(\lambda_\mathcal{S}^{(j)} - \mathbf{R}\lambda_\mathcal{M}^{(i)} - \mathbf{t})'}_{\Delta'_{ij}} \underbrace{(\Sigma_\mathcal{S}^{(j)} + \mathbf{R}\Sigma_\mathcal{M}^{(i)}\mathbf{R}')^{-1}}_{\mathbf{G}_{ij}^{-1}} \underbrace{(\lambda_\mathcal{S}^{(j)} - \mathbf{R}\lambda_\mathcal{M}^{(i)} - \mathbf{t})}_{\Delta_{ij}} \Big\} \\ s.t. \quad &\mathbf{R}(\mathbf{w}) = \exp(J(\mathbf{w}))\mathbf{R}_n, \quad \mathbf{t}(\mathbf{v}) = \mathbf{t}_n + \mathbf{v} \\ &\mathbf{w}, \mathbf{v}, \mathbf{t}, \mathbf{t}_n \in \mathbb{R}^3, \quad \mathbf{R}, \mathbf{R}_n \in SO(3) \\ &\Sigma_\mathcal{M}^{(i)}, \Sigma_\mathcal{S}^{(j)} \text{ are } 3 \times 3 \text{ symmetric positive semidefinite matrics}\end{aligned} \tag{1}$$

the Jacobian vector can be computed as:

$$\frac{\partial \mathcal{O}_{ij}}{\partial \mathbf{v}} = \frac{\partial \mathcal{O}_{ij}}{\partial \Delta_{ij}} \cdot \frac{\partial \Delta_{ij}}{\partial \mathbf{v}_k} = -\mathcal{O}_{ij}\Delta'_{ij}\mathbf{G}_{ij}^{-1}(-1)I_{3,3} = \mathcal{O}_{ij}\Delta'_{ij}\mathbf{G}_{ij}^{-1} \tag{2}$$

$$\frac{\partial \mathcal{O}_{ij}}{\partial \mathbf{w}} = \frac{\partial |\mathbf{G}_{ij}|^{-1/2}}{\partial \mathbf{w}} \cdot \exp(-\frac{1}{2}\Delta'_{ij}\mathbf{G}_{ij}^{-1}\Delta_{ij}) + |\mathbf{G}_{ij}|^{-1/2} \cdot \frac{\partial \exp(-\frac{1}{2}\Delta'_{ij}\mathbf{G}_{ij}^{-1}\Delta_{ij})}{\partial \mathbf{w}} \tag{3}$$

the term $\frac{\partial|\mathbf{G}_{ij}|^{-1/2}}{\partial\mathbf{w}}$ in (3) is:

$$\begin{aligned}\frac{\partial|\mathbf{G}_{ij}|^{-1/2}}{\partial\mathbf{w}} &= \frac{\partial|\mathbf{G}_{ij}|^{-1/2}}{\partial|\mathbf{G}_{ij}|}\frac{\partial|\mathbf{G}_{ij}|}{\partial\mathbf{G}_{ij}}\frac{\partial\mathbf{G}_{ij}}{\partial\mathbf{w}} \\ &= -\frac{1}{2}|\mathbf{G}_{ij}|^{-3/2}\cdot|\mathbf{G}_{ij}|\mathrm{vec}(\mathbf{G}'_{ij})'\cdot\frac{\partial\mathbf{G}_{ij}}{\partial\mathbf{w}} \\ &= -\frac{1}{2}|\mathbf{G}_{ij}|^{-1/2}\mathrm{vec}(\mathbf{G}_{ij})'\cdot\frac{\partial\mathbf{G}_{ij}}{\partial\mathbf{w}}\end{aligned}\tag{4}$$

and the term $\frac{\partial\exp(-\frac{1}{2}\Delta'_{ij}\mathbf{G}_{ij}^{-1}\Delta_{ij})}{\partial\mathbf{w}}$ of (3) is:

$$\begin{aligned}\frac{\partial\exp(-\frac{1}{2}\Delta'_{ij}\mathbf{G}_{ij}^{-1}\Delta_{ij})}{\partial\mathbf{w}} &= \frac{\partial\exp(-\frac{1}{2}\Delta'_{ij}\mathbf{G}_{ij}^{-1}\Delta_{ij})}{\partial(\Delta'_{ij}\mathbf{G}_{ij}^{-1}\Delta_{ij})}\cdot\frac{\partial(\Delta'_{ij}\mathbf{G}_{ij}^{-1}\Delta_{ij})}{\partial\mathbf{w}} \\ &= -\frac{1}{2}\exp(-\frac{1}{2}\Delta'_{ij}\mathbf{G}_{ij}^{-1}\Delta_{ij})\frac{\partial(\Delta'_{ij}\mathbf{G}_{ij}^{-1}\Delta_{ij})}{\partial\mathbf{w}}\end{aligned}\tag{5}$$

and

$$\frac{\partial\left(\Delta'_{ij}\mathbf{G}_{ij}^{-1}\Delta_{ij}\right)}{\partial\mathbf{w}} = \frac{\partial\left[\Delta'_{ij}\left(\mathbf{G}_{ij}^{-1}\Delta_{ij}\right)\right]}{\partial\mathbf{w}} = \Delta'_{ij}\frac{\partial(\mathbf{G}_{ij}^{-1}\Delta_{ij})}{\partial\mathbf{w}} + (\Delta'_{ij}\mathbf{G}_{ij}^{-1})\frac{\partial\Delta_{ij}}{\partial\mathbf{w}}\tag{6}$$

where term $\frac{\partial(\mathbf{G}_{ij}^{-1}\Delta_{ij})}{\partial\mathbf{w}}$ will be computed as:

$$\begin{aligned}\frac{\partial(\mathbf{G}_{ij}^{-1}\Delta_{ij})}{\partial\mathbf{w}} &= \left(\Delta'_{ij}\otimes I_{3,3}\right)\frac{\partial\mathbf{G}_{ij}^{-1}}{\partial\mathbf{w}} + \left(I_{1,1}\otimes\mathbf{G}_{ij}^{-1}\right)\frac{\partial\Delta_{ij}}{\partial\mathbf{w}} \\ &= \left(\Delta'_{ij}\otimes I_{3,3}\right)\frac{\partial\mathbf{G}_{ij}^{-1}}{\partial\mathbf{G}_{ij}}\frac{\partial\mathbf{G}_{ij}}{\partial\mathbf{w}} + \left(I_{1,1}\otimes\mathbf{G}_{ij}^{-1}\right)\frac{\partial\Delta_{ij}}{\partial\mathbf{w}} \\ &= \left(\Delta'_{ij}\otimes I_{3,3}\right)\left[-(G_{ij}^{-1}\otimes G_{ij}^{-1})\right]\frac{\partial\mathbf{G}_{ij}}{\partial\mathbf{w}} + \left(I_{1,1}\otimes\mathbf{G}_{ij}^{-1}\right)\frac{\partial\Delta_{ij}}{\partial\mathbf{w}} \\ &= -\left(\Delta'_{ij}G_{ij}^{-1}\otimes G_{ij}^{-1}\right)\frac{\partial\mathbf{G}_{ij}}{\partial\mathbf{w}} + \mathbf{G}_{ij}^{-1}\frac{\partial\Delta_{ij}}{\partial\mathbf{w}}\end{aligned}\tag{7}$$

and then we can compute $\frac{\partial\Delta_{ij}}{\partial\mathbf{w}}$ which appears in (6)(7) as:

$$\begin{aligned}\frac{\partial\Delta_{ij}}{\partial\mathbf{w}} &= \frac{\partial\left[-\exp(J(\mathbf{w}))\mathbf{R}_n\lambda_{\mathcal{M}}^{(i)}\right]}{\partial\exp(J(\mathbf{w}))}\cdot\frac{\partial\exp(J(\mathbf{w}))}{\partial J(\mathbf{w})}\cdot\frac{\partial J(\mathbf{w})}{\partial\mathbf{w}} \\ &= -(\lambda_{\mathcal{M}}^{(i)\prime}\mathbf{R}'_n\otimes I_{3,3})\cdot I_{9,9}\cdot\frac{\partial J(\mathbf{w})}{\partial\mathbf{w}} \\ &= J(\mathbf{R}_n\lambda_{\mathcal{M}}^{(i)})\end{aligned}\tag{8}$$

the term $\frac{\partial\mathbf{G}}{\partial\mathbf{w}}$ which appears in (4)(6) is computed:

$$\begin{aligned}\frac{\partial\mathbf{G}_{ij}}{\partial\mathbf{w}} &= \frac{\partial\mathbf{G}_{ij}}{\partial\mathbf{R}}\frac{\partial\mathbf{R}}{\partial\mathbf{w}} = \frac{\partial(\mathbf{R}\Sigma_{\mathcal{M}}^{(i)}\mathbf{R}')}{\partial\mathbf{R}}\frac{\partial\mathbf{R}}{\partial\mathbf{w}} = \frac{\partial\Big((\mathbf{R}(\mathbf{K}\mathbf{K}')\mathbf{R}'\Big)}{\partial\mathbf{R}}\frac{\partial\mathbf{R}}{\partial\mathbf{w}} \\ &= \frac{\partial\Big((\mathbf{R}\mathbf{K})(\mathbf{R}\mathbf{K})'\Big)}{\partial\mathbf{R}\mathbf{K}}\cdot\frac{\partial\mathbf{R}\mathbf{K}}{\partial\mathbf{R}}\cdot\frac{\partial\mathbf{R}}{\partial\mathbf{w}} \\ &= (I_{9,9}+T_{3,3})(\mathbf{R}\mathbf{K}\otimes I_{3,3})\cdot(\mathbf{K}'\otimes I_{3,3})\cdot\frac{\partial\mathbf{R}}{\partial\mathbf{w}} \\ &= (I_{9,9}+T_{3,3})(\mathbf{R}\mathbf{K}\mathbf{K}'\otimes I_{3,3})\cdot\frac{\partial\mathbf{R}}{\partial\mathbf{w}} \\ &= (I_{9,9}+T_{3,3})(\mathbf{R}\boldsymbol{\Sigma}_{\mathcal{M}}^{(i)}\otimes I_{3,3})\cdot\frac{\partial\mathbf{R}}{\partial\mathbf{w}}\end{aligned}\tag{9}$$

where vectorization operator $\mathbf{vec}(\cdot)$ and $T_{n,n}$ is defined in (Facker, 2004). Finally, $\frac{\partial\mathbf{R}}{\partial\mathbf{w}}$ is computed:

$$\frac{\partial\mathbf{R}}{\partial\mathbf{w}} = \frac{\partial\left[\exp(J(\mathbf{w})\mathbf{R}_n)\right]}{\partial\exp(J(\mathbf{w}))}\cdot\frac{\partial\exp(J(\mathbf{w}))}{\partial J(\mathbf{w})}\cdot\frac{\partial J(\mathbf{w})}{\partial\mathbf{w}} = (\mathbf{R}'_n\otimes I_{3,3})\cdot\frac{\partial J(\mathbf{w})}{\partial\mathbf{w}}\tag{10}$$

thus we can rewrite the entries of Jacobian vector by substituting (4)(6)(7)(8)(9)(10) into (2)(3) :

$$\frac{\partial\mathcal{O}_{ij}}{\partial\mathbf{v}} = \mathcal{O}_{ij}\Delta'_{ij}\mathbf{G}_{ij}^{-1}\tag{11}$$

$$\begin{aligned}\frac{\partial \mathcal{O}_{ij}}{\partial \mathbf{w}} = \quad & \Big\{ \Big(\tfrac{1}{2} (\Delta'_{ij} \mathbf{G}_{ij}^{-1}) \otimes (\Delta'_{ij} \mathbf{G}_{ij}^{-1}) - \mathbf{vec}(\mathbf{G}_{ij}^{-1})' \Big) \\ & \cdot \tfrac{\partial \mathbf{G}_{ij}}{\partial \mathbf{w}} - \Delta'_{ij} J(\mathbf{R}_n \lambda_m^{(i)}) \Big\} \mathcal{O}_{ij}\end{aligned} \tag{12}$$

where

$$\frac{\partial \mathbf{G}_{ij}}{\partial \mathbf{w}} = (\mathbf{I}_9 + T_{3,3})(\mathbf{R} \boldsymbol{\Sigma}_{\mathcal{M}}^{(i)} \boldsymbol{\Sigma}' \otimes \mathbf{I}_3) \frac{\partial J(\mathbf{w})}{\partial \mathbf{w}} \tag{13}$$

By summing up gradients computed on all pairs, we can get:

$$\frac{\partial \mathbf{O}}{\partial \mathbf{v}} = \sum_{i=1,j=1}^{m_{\mathcal{M}}, m_{\mathcal{S}}} \frac{1}{(2\pi)^{3/2}} \frac{\partial \mathcal{O}_{ij}}{\partial \mathbf{v}} \qquad \frac{\partial \mathbf{O}}{\partial \mathbf{w}} = \sum_{i=1,j=1}^{m_{\mathcal{M}}, m_{\mathcal{S}}} \frac{1}{(2\pi)^{3/2}} \frac{\partial \mathcal{O}_{ij}}{\partial \mathbf{w}} \tag{14}$$

References

Facker, P. (2004, April). Notes on Matrix Calculus.